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Contributions of

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to the "Concise Encyclopedia of Supersymmetry"

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Abstract

The present paper gathers our contributions to the supersymmetry encyclopedia edited by J. Bagger, S. Duplij and W. Siegel. The first part (by M. Daoud and M. Kibler) deals with the concept of k-fermions, i.e., objects interpolating between bosons and fermions. The second part (by F. Gieres) is devoted to geometric structures on two-dimensional surfaces, as appearing for instance in conformal models underlying (super) string theory as well as in integrable field theories.

Contents

1 ON FRACTIONAL SUPERSYMMETRIC QUANTUM MECHANICS

1.1 k-FERMIONS

k-FERMIONS, a set of objects interpolating between fermions and bosons. A k-fermionic algebra F_k is spanned by five operators f_-, f_+, f_+^+, f_-^+ and N through the following relations: (i) The (f_-, f_+, N) relations $f_- f_+ - q f_+ f_- = 1$, $(f_-)^k = (f_+)^k = 0$, $N f_- - f_- N = -f_-$ and $N f_+ - f_+ N = f_+$ correspond to a q-uon algebra. (ii) The (f_+^+, f_-^+, N) relations $f_+^+ f_-^+ - \bar{q} f_-^+ f_+^+ = 1$, $\left(f_+^+\right)^k = \left(f_-^+\right)^k = 0$, $N f_+^+ - f_+^+ N = -f_+^+$ and $N f_-^+ - f_-^+ N = f_-^+$ correspond to a $\bar{q}$-uon algebra. (iii) The (f_-, f_+, f_+^+, f_-^+) relations $f_- f_+^+ - q^{-\frac{1}{2}} f_+^+ f_- = 0$ and $f_+ f_-^+ - q^{+\frac{1}{2}} f_-^+ f_+ = 0$ reflect the fact that the q-uon and $\bar{q}$-uon algebras do not commute. The number $q := \exp(2\pi \mathrm{i}/k)$, with $k \in \mathbf{N} \setminus \{0, 1\}$, is a root of unity and $\bar{q}$ denotes the complex conjugate of q. The couple (f_-, f_+^+) of annihilation operators is connected to the couple (f_+, f_-^+) of creation operators via the Hermitean conjugation relations $f_+^+ = (f_+)^\dagger$ and $f_-^+ = (f_-)^\dagger$; furthermore, N is an Hermitean operator. The case $k = 2$ corresponds to ordinary fermions and the case $k \to \infty$ to ordinary bosons. In the two latter cases, one can take $f_- \equiv f_+^+$ and $f_+ \equiv f_-^+$. In the other cases, the consideration of the two couples (f_-, f_+^+) and (f_+, f_-^+) is necessary. In the case where k is arbitrary, one speaks of k-fermions. Obviously, the k-fermions share some common properties with anyons although they are not restricted to live in a two-dimensional space.

A k-dimensional linear representation of the algebra F_k, on a k-dimensional Hilbert space spanned by the orthonormal set $\{|n\rangle : n = 0, 1, \cdots, k-1\}$, is easily obtained from

$$f_- |n\rangle = \left(\left[n + s - \frac{1}{2} \right]_q \right)^{\frac{1}{2}} |n-1\rangle \quad \text{with} \quad f_- |0\rangle = 0 \tag{1}$$

$$f_+ |n\rangle = \left(\left[n + s + \frac{1}{2} \right]_q \right)^{\frac{1}{2}} |n+1\rangle \quad \text{with} \quad f_+ |k-1\rangle = 0 \tag{2}$$

$$f_+^+ |n\rangle = \left(\left[n + s - \frac{1}{2} \right]_{\bar{q}} \right)^{\frac{1}{2}} |n-1\rangle \quad \text{with} \quad f_+^+ |0\rangle = 0 \tag{3}$$

$$f_-^+ |n\rangle = \left(\left[n + s + \frac{1}{2} \right]_{\bar{q}} \right)^{\frac{1}{2}} |n+1\rangle \quad \text{with} \quad f_-^+ |k-1\rangle = 0 \tag{4}$$

and $N|n\rangle = n|n\rangle$. Here, $s := 1/2$ and $[x]_p := (1 - p^x)/(1 - p)$, for $x \in \mathbf{R}$, where $p = q, \bar{q}$; in the following, the factorial $[n]_p!$ is defined by $[n]_p! := [1]_p [2]_p \cdots [n]_p$ for $n \in \mathbf{N}^*$ and $[0]_p! := 1$ with $p = q, \bar{q}$.

It is possible to find a realization of the operators f_-, f_+, f_+^+ and f_-^+ in Eqs. (1)-(4) in terms of Grassmann variables $(\theta, \bar{\theta})$ and their q- and $\bar{q}$-derivatives $(\partial_\theta, \partial_{\bar{\theta}})$. The Grassmann variables θ and $\bar{\theta}$ are such that $\theta^k = \bar{\theta}^k = 0$. The sets $\{1, \theta, \cdots, \theta^{k-1}\}$ and $\{1, \bar{\theta}, \cdots, \bar{\theta}^{k-1}\}$ span two isomorphic Grassmann algebras. The q- and $\bar{q}$-derivatives are defined by

$$\partial_\theta f(\theta) := \frac{f(q\theta) - f(\theta)}{(q-1)\theta}, \quad \partial_{\bar{\theta}} g(\bar{\theta}) := \frac{g(\bar{q}\bar{\theta}) - g(\bar{\theta})}{(\bar{q}-1)\bar{\theta}} \tag{5}$$

Therefore, by taking the Grassmanian realization $f_+ = \theta$, $f_- = \partial_\theta$, $f_-^+ = \bar{\theta}$ and $f_+^+ = \partial_{\bar{\theta}}$, one has $\partial_\theta \theta - q\theta\partial_\theta = 1$, $(\partial_\theta)^k = \theta^k = 0$, $\partial_{\bar{\theta}}\bar{\theta} - \bar{q}\bar{\theta}\partial_{\bar{\theta}} = 1$, $(\partial_{\bar{\theta}})^k = \bar{\theta}^k = 0$, $\partial_\theta\partial_{\bar{\theta}} - q^{-\frac{1}{2}}\partial_{\bar{\theta}}\partial_\theta = 0$ and $\theta\bar{\theta} - q^{+\frac{1}{2}}\bar{\theta}\theta = 0$, modulo Eq. (5). Following Majid and Rodríguez-Plaza (see [1,2]), it is useful to define the integration process

$$\int d\theta \, \theta^n = \int d\bar{\theta} \, \bar{\theta}^n := 0 \quad \text{for} \quad n = 0, 1, \cdots, k-2 \tag{6}$$

and

$$\int d\theta \, \theta^{k-1} = \int d\bar{\theta} \, \bar{\theta}^{k-1} := 1 \tag{7}$$

In the particular case $k = 2$, Eqs. (6) and (7) describe the Berezin integration for ordinary Grassmann variables.

The states

$$|\theta) := \sum_{n=0}^{k-1} \frac{\theta^n}{([n]_q!)^{\frac{1}{2}}} |n\rangle, \quad |\bar{\theta}) := \sum_{n=0}^{k-1} \frac{\bar{\theta}^n}{([n]_{\bar{q}}!)^{\frac{1}{2}}} |n\rangle \tag{8}$$

are finite linear combinations of the eigenvectors $|n\rangle$ of the operator N. The states (8) are k-fermionic coherent states in the sense that they satisfy the eigenvalue equations $f_-|\theta) = \theta|\theta)$ and $f_+^+|\bar{\theta}) = \bar{\theta}|\bar{\theta})$. Similarly, one can construct the k-fermionic coherent states $(\theta|$ and $(\bar{\theta}|$ as the dual states of the coherent states $|\theta)$ and $|\bar{\theta})$, respectively. The coherence factor $g^{(m)}$ of order m is defined by

$$g^{(m)} := \frac{\left(\theta\right| \left(f_-^+\right)^m (f_-)^m \left|\theta\right)}{\left(\theta|f_-^+ f_-|\theta\right)^m} \tag{9}$$

As an interesting result, Eq. (9) yields $\left|g^{(m)}\right| = 0$ for $m > k-1$ and $\left|g^{(m)}\right| = 1$ for $m \leq k-1$. From this result it is concluded that, in a many-particle scheme, a given k-fermionic quantum state of fractional spin $S = 1/k$ cannot be occupied by more than $k-1$ identical k-fermions. This statement induces a generalized Pauli exclusion principle.

A pair of Q-uons (with Q generic) can give rise to a pair of bosons and a pair of q-uons (with q a root of unity) by making use of a limiting procedure where $Q \to q$. This is quite well-known in the case of the Macfarlane or Biedenharn

Q-uons. The limiting procedure can be adapted to the case of the Arik and Coon Q-uons in the following way. One begins with a pair of Q-uons (a_-, a_+) satisfying the relation $a_-a_+ - Qa_+a_- = 1$ where Q is generic. One assumes that $Q \to q := \exp(2\pi\mathrm{i}/k)$, with $k \in \mathbf{N} \setminus \{0, 1\}$, and then one takes $(a_\pm)^k = 0$ for $q = Q$. If one defines

$$b_\pm := \lim_{Q \to q} \frac{(a_\pm)^k}{([k]_Q!)^{\frac{1}{2}}} \tag{10}$$

one obtains the result $b_-b_+ - b_+b_- = 1$ so that the pair (b_-, b_+) is a pair of ordinary bosons. One redefines $a_\pm$ as $f_\pm$ for $Q = q$. Therefore, one also has a pair of k-fermions (f_-, f_+) satisfying $f_-f_+ - qf_+f_- = 1$. It can be proved that the b's commute with the f's. As a conclusion, the set $\{b_-, b_+, f_-, f_+\}$ is entirely generated from the set $\{a_-, a_+\}$. Indeed, the decomposition Q-uon $\to$ boson + k-fermion or more precisely $\{a_-, a_+\} \to \{b_-, b_+, f_-, f_+\}$, based on (10), corresponds to the Z-line $\leftrightarrow$ (z, θ)-superspace isomorphism described by Dunne *et al.* and Mansour *et al.* (see [1,2]).

What happens to an ordinary Q-deformed coherent state

$$|Z) := \sum_{n=0}^{\infty} \frac{Z^n}{([n]_Q!)^{\frac{1}{2}}} \, |n\rangle \tag{11}$$

(with Q generic and Z a complex number) when Q goes to a root of unity ? By using the just described decomposition $Z \leftrightarrow (z, \theta)$, it is possible to show that the limit

$$|z, \theta) := \lim_{Z \to (z,\theta)} \lim_{Q \to q} |Z) \tag{12}$$

is the product

$$|z, \theta) = \sum_{r=0}^{\infty} \frac{z^r}{\sqrt{r!}} \, |r\rangle \otimes \sum_{s=0}^{k-1} \frac{\theta^s}{([s]_q!)^{\frac{1}{2}}} \, |s\rangle \tag{13}$$

of an ordinary bosonic coherent state (z is a bosonic complex variable) and a k-fermionic coherent state (θ is a k-fermionic Grassmann variable). The state $|z, \theta)$ defined through (11)-(13) is called a fractional supercoherent state. Note that $|z, \theta)$ is an eigenstate of the operator b_-f_- with the eigenvalue $z\theta$. Furthermore, one can generate $|z, \theta)$ from the vacuum state $|0\rangle \otimes |0\rangle \equiv |r = 0\rangle \otimes |s = 0\rangle$ owing to the operator $D_q(z, \theta) := \exp(zb_+) \, \mathrm{e}_q(\theta f_+)$ where e_q stands for a q-deformed exponential. As a matter of fact, one has $|z, \theta) = D_q(z, \theta) \, |0\rangle \otimes |0\rangle$ and thus the operator $D_q(z, \theta)$ plays the rôle of a displacement or dilation operator.

A legitimate question arises: what is the Hamiltonian H having the fractional supercoherent states $|z, \theta)$ as coherent states ? An immediate answer can be obtained in the case $k = 2$. In this case, the state $|z, \theta)$ turns out to be a supercoherent state for an ordinary supersymmetric oscillator. Such a supersymmetric oscillator corresponds to a Z_2-grading. Since the fractional supercoherent state $|z, \theta)$ corresponds to a Z_k-grading, it is expected that the Hamiltonian H is the one for a fractional supersymmetric oscillator corresponding to a Z_k-grading. This Hamiltonian can be constructed as follows. The basic ingredients consist of a pair of ordinary bosons (b_-, b_+) and a pair of k-fermions (f_-, f_+). The f's satisfy q-commutation relations and the b's usual commutation relations (see above). In addition, the f's commute with the b's. Indeed, the pairs (b_-, b_+) and (f_-, f_+) may be considered as originating from a pair of Q-uons (a_-, a_+) through the isomorphism between the braided line and the one-dimensional superspace. One defines the operators X_- and X_+ by

$$X_- := b_- \left[f_- + \frac{(f_+)^{k-1}}{[k-1]_q!} \right], \quad X_+ := b_+ \left[f_- + \frac{(f_+)^{k-1}}{[k-1]_q!} \right]^{k-1} \tag{14}$$

and the operator K by

$$K := f_- f_+ - f_+ f_- \tag{15}$$

It is a simple matter of calculation to check that X_-, X_+ and K satisfy

$$X_- X_+ - X_+ X_- = 1, \quad K^k = 1 \tag{16}$$

$$K X_+ - q X_+ K = 0, \quad K X_- - \bar{q} X_- K = 0 \tag{17}$$

plus some ordinary commutation relations with the bilinear form $M := X_+ X_-$, namely

$$M X_- - X_- M = -X_-, \quad M X_+ - X_+ M = X_+, \quad MK - KM = 0 \tag{18}$$

The operators X_-, X_+, K and M thus generate the extended Weyl-Heisenberg algebra (16)-(18). The form of the usual commutation relation $[X_-, X_+] = 1$ is the same as for the ordinary Weyl-Heisenberg algebra ; it differs from the one used by Plyushchay and generalized by Quesne and Vansteenkiste (see [2]). The next step is to introduce the k projection operators

$$\Pi_i := \frac{1}{k} \sum_{s=0}^{k-1} q^{si} K^s, \quad i = 0, 1, \cdots, k-1 \tag{19}$$

for the cyclic group Z_k. One is thus in a position to define the two supercharges

$$Q_- := X_-(1 - \Pi_{k-1}), \quad Q_+ := X_+(1 - \Pi_0) \tag{20}$$

among k possible definitions. It is easily verified that the Q's defined by (14)-(15) and (19)-(20) satisfy the nilpotency relations $(Q_-)^k = (Q_+)^k = 0$. Following the technique developed by Rubakov and Spiridonov in their work on para-fermions (see [2]), the Hamiltonian H can be introduced by means of the defining relation

$$(Q_-)^{k-1} Q_+ + (Q_-)^{k-2} Q_+ Q_- + \cdots + Q_+ (Q_-)^{k-1} = (Q_-)^{k-2} H \tag{21}$$

Equation (21) leads to an expression of the operator H which satisfies the commutation relation $HQ_\pm - Q_\pm H = 0$ and thus the two supercharges Q_- and Q_+ can be regarded as constants of motion. It can be shown that the case $k = 2$ corresponds to the well-known supersymmetric harmonic oscillator of the Z_2-graded supersymmetric quantum mechanics. In the case where k is arbitrary, the Hamiltonian H defines a fractional or Z_k-graded supersymmetric oscillator.

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1.2 k-FERMIONIC COHERENT STATE

k-FERMIONIC COHERENT STATE, an extension of the notions of bosonic coherent state and fermionic coherent state. A k-fermionic coherent state is a linear combination of the eigenvectors of a generalized number operator N, a generator of a k-fermionic algebra $\{f_-, f_+, f_+^+, f_-^+, N\}$, with coefficients in a Grassmann algebra. Such a state is an eigenvector of an annihilation operator (f_- or f_+^+). The k-fermionic algebra $\{f_-, f_+, f_+^+, f_-^+, N\}$, with $k \in \mathbf{N} \setminus \{0, 1\}$, results from the combination of a q-uon algebra $\{f_-, f_+, N\}$ and a $\bar{q}$-uon algebra $\{f_+^+, f_-^+, N\}$ with $q := \exp(2\pi\mathrm{i}/k)$ and $q\bar{q} = 1$. The q-uon algebra and the $\bar{q}$-uon algebra do not commute except in the cases $k = 2$ and $k \to \infty$ for which they coincide ($f_- \equiv f_+^+$ and $f_+ \equiv f_-^+$). Therefore, the k-fermionic coherent state is an ordinary fermionic coherent state for $k = 2$ and an ordinary bosonic coherent state for $k \to \infty$. A coherence factor, that generealizes the coherence factor used in quantum optics, can be defined from the expectation values on a k-fermionic coherent state of the operators $(f_-^+)^m (f_-)^m$ and $f_-^+ f_-$ with $m \in \mathbf{N} \setminus \{0\}$. The value of this generalized coherence factor shows that a state of fractional spin $1/k$ cannot be occupied by more than $k - 1$ identical k-fermions.

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1.3 FRACTIONAL SUPERCOHERENT STATE

FRACTIONAL SUPERCOHERENT STATE, a generalization of the notion of supercoherent state associated to a superoscillator. A fractional (or Z_k-graded with $k \in \mathbf{N}\setminus\{0, 1\}$) supercoherent state can be defined as the product of a bosonic coherent state by a k-fermionic coherent state. In other words, it corresponds to the decomposition of a Q-uon with Q generic in terms of a boson and a k-fermion, i.e., to the Z-line $\leftrightarrow$ (z, θ)-superspace isomorphism [1] (with z a complex number and θ a Grassmann variable). From a pratical point of view, it may be obtained as the limit of a Q-deformed (with Q generic) coherent state $|Z)$ when

$Z \to (z, \theta)$ and $Q \to \exp(2\pi \mathrm{i}/k)$ with $k \in \mathbf{N} \setminus \{0, 1\}$. The so-obtained state is an eigenstate of the operator $b_- f_-$ where b_- is a bosonic annihilation operator and f_- a k-fermionic annihilation operator. It can be also generated by acting on the vacuum state with a displacement operator. In the special case $k = 2$, a fractional supercoherent state involves only a bosonic coherent state and a fermionic coherent state as described in [2] in connection with the ordinary or Z_2-graded supersymmetric oscillator. A fractional supercoherent state corresponding to k arbitrary is a coherent state for a fractional or Z_k-graded supersymmetric oscillator [3].

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1.4 ON THE FRACTIONAL SUPERSYMMETRIC OSCILLATOR

FRACTIONAL SUPERSYMMETRIC OSCILLATOR, a generalization of the supersymmetric oscillator occurring in ordinary or Z_2-graded supersymmetric quantum mechanics. A fractional or Z_k-graded supersymmetric oscillator corresponds to an extended Weyl-Heisenberg algebra spanned by three operators X_-, X_+ and K which can be constructed from a pair of ordinary bosons and a pair of k-fermions. The extended Weyl-Heisenberg algebra is defined by $X_- X_+ - X_+ X_- = 1$, $K X_+ = q X_+ K$, $K X_- = \bar{q} X_- K$ and $K^k = 1$ with $q = \exp(2\pi \mathrm{i}/k)$, $k \in \mathbf{N} \setminus \{0, 1\}$, and $q\bar{q} = 1$. The operators X_-, X_+ and K can be used for defining two supercharges Q_- and Q_+ which satisfy $(Q_-)^k = (Q_+)^k = 0$. The latter nilpotency relations are the signature of a Z_k-grading. The multilinear form $(Q_-)^{k-1} Q_+ + (Q_-)^{k-2} Q_+ Q_- + \cdots + Q_+ (Q_-)^{k-1} = (Q_-)^{k-2} H$ serves to define the Hamiltonian H, that commutes with Q_- and Q_+, of a Z_k-graded supersymmetric oscillator in a way similar to the one used by Rubakov and Spiridonov for para-fermions [1]. The case $k = 2$ yields the ordinary supersymmetric oscillator for which the spectrum consists of equally spaced levels, the ground state being a singlet and all the excited states being doublets. The spectrum of the Z_k-graded supersymmetric oscillator contains equally spaced levels with, in increasing order, one singlet, one doublet, $\cdots$, one $(k-1)$-plet and an infinite sequence of k-plets. The Z_k-graded supersymmetric oscillator admits fractional or Z_k-graded supercoherent states [2]. Note that there exists other extensions of the Weyl-Heisenberg algebra which lead to other spectra [3].

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2 RIEMANN SURFACES

2.1 BELTRAMI DIFFERENTIALS

General references : [1,2,3,4]

Riemann surface. A Riemann surface $\mathbf{\Sigma}$ is a connected topological 2-manifold which is equipped with a complex structure (or, equivalently, it is a smooth connected 2-manifold which is oriented and equipped with a conformal class of metrics – see below). In essence, this means that any two systems of local complex coordinates, say z and z', are related by a conformal (=bi-holomorphic) coordinate transformation, $z \stackrel{conf.}{\longrightarrow} z'(z)$. If the Riemann surface $\mathbf{\Sigma}$ is compact, it is diffeomorphic to a two-sphere with a certain number of handles, the so-called genus $g \geq 0$ of the surface.

Conformal classes of metrics and Beltrami differentials. Consider a Riemannian 2-manifold (M, g), i.e. a real, smooth 2-manifold M which is equipped with a positive-definite metric g. Using the complex notation $dz = dx + idy$ the line element associated to the metric can always be written as

$$ds^2 = |\rho|^2 |dz + d\bar{z}\, \mu_{\bar{z}}{}^z|^2 \ . \tag{1}$$

Here, ρ and $\mu \equiv \mu_{\bar{z}}{}^z$ are smooth complex-valued functions of $z, \bar{z}$ and the positive-definiteness of the metric is expressed by the condition $|\mu| < 1$. The function ρ is usually called the conformal factor and μ the Beltrami coefficient (or Beltrami differential) while expression (1) is known as the Beltrami parametrization of the metric. Weyl transformations of the metric amount to a rescaling of the factor ρ by a smooth complex-valued function. Thus, the coefficient μ is Weyl-inert and parametrizes conformal classes of metrics.

Complex structures. To parametrize complex structures, one compares the Beltrami parametrization of the metric, $ds^2 \propto |dz + d\bar{z}\, \mu|^2$, with the metric written in terms of isothermal coordinates, $ds^2 \propto |dZ|^2$ and argues as follows. If $(z, \bar{z})$ denotes a reference system of holomorphic coordinates (corresponding to a reference complex structure), then the holomorphic coordinates $(Z, \bar{Z})$ corresponding to the complex structure parametrized by μ are given by the relation

$$dZ = \lambda(z, \bar{z})[dz + d\bar{z}\, \mu(z, \bar{z})]$$

(as well as the complex conjugate relation). Here, λ and μ are smooth complex-valued functions and the condition $d\,(dZ) = 0$ is equivalent to λ satisfying the differential equation

$$(\bar{\partial} - \mu\partial)\lambda = (\partial\mu)\lambda \ ,$$

where $\partial \equiv \partial_z \equiv \partial/\partial z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. Thus, λ is to be viewed as an integrating factor for the structure relation $d(dZ) = 0$. Locally, Z and $\bar{Z}$ can be given by smooth invertible functions $(z,\bar z) \to \Big(Z(z,\bar z), \bar Z(z,\bar z)\Big)$ so that $\lambda = \partial Z$ and $\mu = (\bar\partial Z)/(\partial Z)$. The latter relation is equivalent to $Z(z,\bar z)$ satisfying the Beltrami equation $(\bar\partial - \mu\partial)Z = 0$.

Transformation laws and holomorphic factorization. The Beltrami differential can be viewed as component of the smooth tensor field $\mu_{\bar z}{}^z(z,\bar z) d\bar z \otimes \partial_z$ so that its transformation law under a conformal change of coordinates $z \to z'(z)$ is given by $\mu_{\bar z'}{}^{z'} = (\partial_{\bar z'}\bar z)\,(\partial_z z')\,\mu_{\bar z}{}^z$. Its transformation law under an infinitesimal diffeomorphism $z \to z + \xi(z,\bar z)$ (and CC) reads as

$$\delta\mu = [\bar\partial - \mu\partial + (\partial\mu)]c \quad \text{and CC} \ ,$$

where $c \equiv \xi + \mu\bar\xi$ is known as Becchi's reparametrization of the variables $\xi, \bar\xi$. Since the variation of μ only depends on c (and not on $\bar c$), this parametrization manifestly realizes the property of holomorphic factorization which plays a fundamental role in two-dimensional conformal field theory [5].

2.2 DERIVATIVES AND CONNECTIONS

General references : [2]

Conformal fields. A conformal (or primary) field of weight $k \in \mathbf{Z}/2$ on the Riemann surface $\mathbf{\Sigma}$ is a collection $\{c_k(z,\bar z)\}$ of local complex-valued functions on $\mathbf{\Sigma}$ (one for each coordinate system $(z,\bar z)$), transforming according to

$$c'_k(z',\bar z') \ = \ (\partial z')^{-k}\, c_k(z,\bar z)$$

under a conformal change of coordinates. Thus, c transforms linearly with a certain (integer or half-integer) power of the Jacobian of the change of coordinates. The space of conformal fields of weight k on $\mathbf{\Sigma}$ will be denoted by $\mathcal{F}_k$. (More generally, one can introduce conformal fields which also transform with a certain power of $\bar\partial \bar z'$, the Beltrami coefficient $\mu_{\bar z}{}^z$ being an example, but we will not consider such fields in the sequel.) In the mathematics literature, conformal fields are usually referred to as densities or forms of half-integer degree.

Affine connection. An affine connection on the Riemann surface $\mathbf{\Sigma}$ is a collection $\{\gamma \equiv \gamma_z(z,\bar z)\}$ of local complex-valued functions on $\mathbf{\Sigma}$ which are locally holomorphic (i.e. $\partial_{\bar z}\gamma = 0$) and which transform under a conformal change of coordinates as

$$\gamma_{z'}(z') = (\partial z')^{-1}\,[\gamma_z(z) - \partial_z \ln(\partial z')] \ .$$

The only compact Riemann surfaces which admit a globally defined affine connection are those of genus one.

Covariant derivative. Given an affine connection, one can define a covariant derivative which maps conformal fields into conformal fields:

$$\begin{aligned} \nabla_{(k)} : \mathcal{F}_k &\longrightarrow \mathcal{F}_{k+1} \\ c_k &\longmapsto \nabla_{(k)} c_k \ \equiv \ (\partial_z - k\gamma_z) c_k \ . \end{aligned}$$

In the theory of integrable models, the derivative $\nabla_{(1)} = \partial - \gamma$ is also known as the Fréchet Jacobian of $R \equiv \partial\gamma - \frac{1}{2}\gamma^2$ with respect to γ.

Schwarzian derivative. Let $Z(z, \bar{z})$ be a smooth complex-valued function of the local coordinates. For all $(z, \bar{z})$ with $(\partial Z)(z, \bar{z}) \neq 0$, the Schwarzian derivative of Z is defined by

$$\begin{aligned} S(Z, z) &= \partial^2 \ln \partial Z - \frac{1}{2} (\partial \ln \partial Z)^2 \\ &= \frac{\partial^3 Z}{\partial Z} - \frac{3}{2} \left(\frac{\partial^2 Z}{\partial Z} \right)^2 \ . \end{aligned}$$

If the local coordinates are changed by a conformal transformation $z \to z'(z)$, the derivative S transforms according to the chain rule

$$S(Z, z) = S(Z, z') \, (\partial z')^2 + S(z', z) \ .$$

Projective connection. A projective (or Schwarzian) connection on the Riemann surface $\mathbf{\Sigma}$ is a collection $\{R \equiv R_{zz}(z, \bar{z})\}$ of local complex-valued functions on $\mathbf{\Sigma}$ which are locally holomorphic, (i.e. $\partial_{\bar{z}} R = 0$) and which transform inhomogeneously with the Schwarzian derivative under a conformal change of coordinates:

$$R'(z') = (\partial z')^{-2} \left[R(z) - S(z', z) \right] \ .$$

Such connections exist globally on compact Riemann surfaces of any genus. The difference of two projective connections represents a holomorphic quadratic differential. Thereby, the set of all projective connections on a compact Riemann surface of genus g is an affine space; the set of equivalence classes of such connections has dimension $3g - 3$.

One can also define projective connections which are not holomorphic or assume that they are only holomorphic up to some poles (as in the context of Krichever-Novikov algebras). From the physical point of view, the field R and its complex conjugate represent the components of the energy-momentum tensor in two-dimensional conformal field theory.

If γ represents an affine connection, then $R_{zz} \equiv \partial_z \gamma_z - \frac{1}{2}\gamma_z^2$ is a projective connection. The latter expression has the same form as the Riccati equation or Miura transformation which occur in the theory of integrable systems.

2.3 PROJECTIVE STRUCTURES

General references : [2]

Projective transformation. A change of local complex coordinates $Z \to Z'(Z)$ of a Riemann surface which has the form

$$Z' = \frac{aZ + b}{cZ + d} \quad \text{with } a, b, c, d \in \mathbf{C} \quad \text{and } ad - bc = 1 \ , \tag{2}$$

is called a projective (or Möbius or fractional linear) transformation. Such a transformation is characterized by the following property: the mapping $Z \to Z'(Z)$ is a projective transformation if and only if $S(Z', Z) = 0$.

Projective structure. A projective structure on a Riemann surface $\mathbf{\Sigma}$ is an atlas of local complex coordinates for which all coordinate transformations are projective. Every compact Riemann surface admits such a structure. Moreover, there is a one-to-one correspondence between projective structures and projective connections R_{zz}, described by the relation $R_{zz}(z) \equiv S(Z, z)$ where S denotes the Schwarzian derivative, and where the coordinates Z and z belong, respectively, to a projective and a generic atlas of $\mathbf{\Sigma}$. (Indeed, the previous expression has the correct transformation properties thanks to the chain rule for the Schwarzian derivative and it is locally holomorphic since the change of coordinates $z \to Z(z)$ has this property.)

Quasi-primary field. Let $\mathbf{\Sigma}$ be a compact Riemann surface with a given projective structure. Then, a quasi-primary field of weight $k \in \mathbf{Z}/2$ on $\mathbf{\Sigma}$ is a collection $\{C_k(Z, \bar{Z})\}$ of local complex-valued functions on $\mathbf{\Sigma}$ which transform linearly with the k-th power of the Jacobian under a projective change of coordinates (2):

$$C'_k \ = \ (\, cZ + d\,)^{2k}\, C_k \ .$$

2.4 COVARIANT OPERATORS

General references : [6,7,8]

Conformally covariant operator (CCO). A holomorphic n-th order differential operator on the compact Riemann surface $\mathbf{\Sigma}$ is locally given by $L^{(n)} = \partial^n + a_1^{(n)} \partial^{n-1} + \ldots + a_n^{(n)}$ where the coefficients $a_1^{(n)}, \ldots, a_n^{(n)}$ are locally holomorphic functions on $\mathbf{\Sigma}$. Such an operator is called conformally covariant if it is globally defined and if it maps conformal fields (of some weight $k \in \mathbf{Z}/2$) to conformal fields, i.e. $L^{(n)} : \mathcal{F}_k \to \mathcal{F}_{k+n}$. The latter requirement is equivalent to the one that $L^{(n)}$ transforms according to the following operatorial relation under a conformal change of coordinates $z \to z'(z)$:

$$L^{(n)\,\prime} = (\partial z')^{-(k+n)}\, L^{(n)}\, (\partial z')^k \ \ .$$

This relation implies that $k = \frac{1-n}{2}$ and that the coefficient $a_1^{(n)}$ transforms linearly so that it can be set to zero (and indeed it always is). Moreover, it implies that $a_2^{(n)}$ is a multiple of a projective connection R and that the remaining coefficients $a_3^{(n)}, \dots, a_n^{(n)}$ transform in a more complicated way than R. Thus, a conformally covariant operator (CCO) of order n on $\mathbf{\Sigma}$ is a globally well-defined map

$$L^{(n)} : \mathcal{F}_{\frac{1-n}{2}} \longrightarrow \mathcal{F}_{\frac{1+n}{2}}$$

with the local expression

$$L^{(n)} = \partial^n + a_2^{(n)} \partial^{n-2} + \dots + a_n^{(n)} \quad \text{with} \;\; a_2^{(n)} \; = \; \frac{n(n^2-1)}{12} \, R \; .$$

The following results apply equally well to the case where $\mathbf{\Sigma}$ is a real one-dimensional manifold, in which case the changes of coordinates are diffeomorphisms.

Bol operator. The simplest way to construct CCO's consists of starting from the special coordinate system where $a_2^{(n)} = 0$ (i.e. starting from projective coordinates Z and operators which are Möbius covariant) and then going over to generic local coordinates z by a conformal transformation $Z \stackrel{conf.}{\longrightarrow} z$: the dependence of the operators on the projective structure then translates into a dependence on a projective connection. The most basic example is provided by the n-th order derivative $\partial_Z^n \equiv (\partial / \partial Z)^n$ acting on a quasi-primary field of weight $\frac{1-n}{2}$. Upon passage $Z \stackrel{conf.}{\longrightarrow} z$, the quasi-primary field C_k becomes a primary field c_k, both fields being related by $C_k(Z, \bar{Z}) = (\partial Z)^{-k} c_k(z, \bar{z})$, and the Möbius covariant operator ∂_Z^n becomes a CCO L_n, known as Bol operator [6]:

$$\partial_Z^n C_{\frac{1-n}{2}} = (\partial Z)^{-\frac{1+n}{2}} L_n c_{\frac{1-n}{2}} \quad \text{or} \quad L_n = (\partial Z)^{\frac{1+n}{2}} \left(\frac{1}{\partial Z} \partial \right)^n (\partial Z)^{-\frac{1-n}{2}} \; .$$

Thus, the Bol operator L_n represents the conformally covariant version of the differential operator ∂^n, the simplest cases being given by $L_0 = 1$, $L_1 = \partial$ and

$$\begin{aligned} L_2 &= \partial^2 + \frac{1}{2} R \quad , \quad L_3 = \partial^3 + 2R\partial + (\partial R) \\ L_4 &= \partial^4 + 5R\partial^2 + 5(\partial R)\partial + \frac{3}{2}[(\partial^2 R) + \frac{3}{2} R^2] \; , \end{aligned}$$

where $R_{zz}(z) \equiv S(Z, z)$ represents a projective connection.

A useful result (which holds locally on a generic compact Riemann surface $\mathbf{\Sigma}$ and globally if the genus of $\mathbf{\Sigma}$ is one) is the following factorization of Bol operators. Suppose the projective connection R comes from an affine connection γ, i.e. $R = \partial\gamma - \frac{1}{2}\gamma^2$. Writing $n = 2l + 1$ with $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, the Bol operator L_n is then given by $L_n = \nabla_{(l)} \nabla_{(l-1)} \cdots \nabla_{(-l)}$.

The Bol operators admit a wide range of applications, both in mathematics and physics. In complex analysis, these applications include invariants of differential equations, the theory of Hankel operators and twistor theory. In physics, the basic operator L_2 (which is known as Hill operator) appears in the Lax representation of the KdV equation while L_3 appears in the Poisson brackets for the Virasoro algebra or in the conformal Ward identity. Furthermore, Bol operators appear in the Wess-Zumino-Witten (WZW) model and they are related to the null- or singular vectors in the Verma module of the Virasoro algebra [7].

Other CCO's. A second class of CCO's (which exists for each $n \geq 3$) is given by operators $M^{(n)}_{w_3}, \ldots, M^{(n)}_{w_n}$ which do not only depend on the projective structure, but also, in a linear way, on conformal fields $w_3, \ldots, w_n$. Rather than giving a general formula in terms of projective coordinates [6,7,8], we present the explicit expression for some of these CCO's:

$$
\begin{aligned}
M^{(n)}_{w_n} &= w_n \quad , \quad M^{(n)}_{w_{n-1}} = w_{n-1}\partial + \frac{1}{2}(\partial w_{n-1}) \\
M^{(n)}_{w_{n-2}} &= w_{n-2}[\partial^2 - \frac{1-n}{2}R] + (\partial w_{n-2})\partial + \frac{n-1}{2(2n-3)}[\partial^2 - (n-2)R]w_{n-2} \; .
\end{aligned}
$$

Classification theorem for CCO's. The two classes of CCO's that we just presented already exhaust all cases according to the following classification theorem: Any CCO can be reparametrized in the following way in terms of the projective connection R and $n-2$ conformal fields $w_3, \ldots, w_n$:

$$
L^{(n)} = L_n + M^{(n)}_{w_3} + \ldots + M^{(n)}_{w_n} \; . \tag{3}
$$

The relation between the coefficients $a^{(n)}_3, \ldots, a^{(n)}_n$ and the conformal fields $w_3, \ldots, w_n$ is given by differential polynomials which involve R and this relation is invertible.

The parametrization (3) of $L^{(n)}$ in terms of the energy-momentum tensor and some conformal fields is very helpful for the construction and formulation of W_n-algebras. These operators also appear in the construction of hierarchies of integrable equations, e.g. $L^{(3)} = L_3 + M^{(3)}_{w_3} = L_3 + w_3$ yields the Boussinesq sequence.

Gordan's transvectant and multilinear operators. Apart from the linear CCO's considered up to now, one can introduce multilinear CCO's. There exists a unique bilinear CCO $J(\cdot,\cdot)$ known as Gordan's transvectant. The latter encompasses the CCO's $M^{(n)}_{w_k}$ in the sense that $M^{(n)}_{w_k}c \propto J(w_k, c)$. The bilinear operator $J(\cdot,\cdot)$ as well as higher multilinear CCO's appear in the defining relations of W_n-algebras.

2.5 W-ALGEBRAS

General references : [9,10,7,8]

W_n**-algebra.** The so-called W_n-algebra (with $n \geq 3$), which was introduced by Zamolodchikov [9], is a non-linear generalization of the Virasoro algebra (i.e. the two-dimensional conformal algebra). The variable $a_2^{(n)}$ (corresponding to the energy-momentum tensor in two-dimensional conformal field theory) and the conformal fields w_k (with $3 \leq k \leq n$), which parametrize a general CCO of order n, can be viewed as generators of the classical W_n-algebra [7]. In fact, the coefficient $a_2^{(n)}$, i.e. a projective connection multiplied by the factor $k_n = \frac{1}{12}n(n^2-1)$, generates the classical Virasoro algebra through the Poisson brackets

$$\begin{aligned} \{a_2^{(n)}(z'), a_2^{(n)}(z)\} &= \left[k_n \partial^3 + 2a_2^{(n)}\partial + (\partial a_2^{(n)})\right] \delta(z-z') \\ &= k_n L_3 \delta(z-z') \ , \end{aligned}$$

where L_3 denotes the third-order Bol operator. The Poisson bracket of $a_2^{(n)}$ with w_k expresses the fact that w_k is a conformal field of weight k:

$$\{a_2^{(n)}(z'), w_k(z)\} = \left[(\partial w_k) + k w_k \partial\right] \delta(z-z') \ .$$

For $n = 3 = k$, the brackets for w_3 read as

$$\{w_3(z'), w_3(z)\} = -\frac{1}{6} L_5 \delta(z-z') \ ,$$

where L_5 denotes the fifth-order Bol operator. This expression coincides with the second Poisson brackets for the Boussinesq equation. For generic values of n, and for the central terms appearing in quantum W-algebras, we refer to [7]. The wide range of applications of W_n-algebras in two-dimensional conformal field theory, string theory, quantum gravity, statistical mechanics or integrable models is addressed in [10].

Matrix representation of CCO's and zero curvature construction of W_n**-algebras.** The CCO's L_n and $M_{w_k}^{(n)}$ admit a matrix representation which is related to the principal embedding of the Lie algebra $sl(2)$ into $sl(n)$ [7]. Since $sl(2)$ is the Lie algebra of the Möbius group, this algebraic relationship reflects the fact that the CCO's arise from Möbius covariant operators. We illustrate the matrix representation of $L^{(3)} = L_3 + w_3$ by rewriting the scalar, conformally covariant differential equation

$$0 = L^{(3)} f_3 \equiv \left[\partial^3 + 2R\partial + (\partial R) + w_3\right] f_3 \qquad \text{with } f_3 \in \mathcal{F}_{-\frac{1}{2}}$$

as a system of three first-order differential equations:

$$\vec{0} = (\partial - \mathcal{A})\vec{F} \qquad \text{with } \mathcal{A} = \begin{bmatrix} 0 & -R & -w_3 \\ 1 & 0 & -R \\ 0 & 1 & 0 \end{bmatrix} \ , \quad \vec{F} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \ .$$

The matrix $\mathcal{A}$ can be viewed as the z-component of a two-dimensional gauge connection with values in the Lie algebra $sl(3)$. After supplementing $\mathcal{A}$ with a $\bar{z}$-component, one can derive the W_3-algebra by imposing a zero curvature condition on the connection [11,8].

2.6 BIBLIOGRAPHY

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3 SUPER RIEMANN SURFACES

3.1 SUPER BELTRAMI DIFFERENTIALS

Super Riemann surface. A $N = 1$ super Riemann surface (SRS) $\mathbf{S\Sigma}$ is locally parametrized by coordinates $\mathbf{z} = (z, \theta)$ with z even and θ odd [1,2]. The canonical derivatives (representing the canonical basis of the super tangent space) are given by $(\partial, D = \partial_\theta + \theta\partial)$ where $\partial \equiv \partial/\partial z$ and $\partial_\theta \equiv \partial/\partial\theta$. By definition, any two sets of local coordinates on the SRS, $\mathbf{z}$ and $\mathbf{z}'$, are related by a superconformal transformation $\mathbf{z} \to \mathbf{z}'(\mathbf{z})$, i.e. a transformation satisfying $Dz' = \theta'(D\theta')$. This condition implies $D = (D\theta')D'$ (i.e. the fundamental derivative D transforms linearly, just as ∂ does in the non-supersymmetric case) and $(D\theta')(D'\theta) = 1$.

The canonical basis of the super cotangent space is given by the super 1-forms

$$e^z \equiv dz + \theta d\theta \ , \quad e^\theta \equiv d\theta \ ,$$

which satisfy the structure relations $de^z = -e^\theta e^\theta$, $de^\theta = 0$.

Supercomplex structures and super Beltrami differentials. An atlas of superconformal coordinates on $\mathbf{S\Sigma}$ defines a supercomplex structure or, equivalently, a superconformal class of vielbein fields. These structures are parametrized by super Beltrami coefficients $H_{\bar\theta}{}^z, H_\theta{}^z$ (as well as the complex conjugate variables) all of which are odd superfields. More precisely [3], the parametrization is described by choosing a reference coordinate system, denoted by small coordinates $(z, \bar z, \theta, \bar\theta)$, and expressing the canonical 1-forms of the coordinate system $(Z, \bar Z, \Theta, \bar\Theta)$ with respect to the corresponding 1-forms of the reference coordinate system:

$$\begin{aligned} e^Z &= \left[e^z + e^{\bar z} H_{\bar z}{}^z + e^\theta H_\theta{}^z + e^{\bar\theta} H_{\bar\theta}{}^z\right] \Lambda_z{}^Z \\ e^\Theta &= \left[e^z \ + \ e^{\bar z} H_{\bar z}{}^z + e^\theta H_\theta{}^z + e^{\bar\theta} H_{\bar\theta}{}^z\right] \tau_z{}^\Theta \\ &\qquad + \left[e^\theta H_\theta{}^\theta + e^{\bar z} H_{\bar z}{}^\theta + e^{\bar\theta} H_{\bar\theta}{}^\theta\right] \sqrt{\Lambda_z{}^Z} \end{aligned}$$

(and CC). Locally, the coefficients $H_{\bar\theta}{}^z$, $H_\theta{}^z$ take the form

$$H_{\bar\theta}{}^z = \frac{\bar D Z - \Theta \bar D \Theta}{\partial Z + \Theta\partial\Theta} \ , \quad H_\theta{}^z = \frac{DZ - \Theta D\Theta}{\partial Z + \Theta\partial\Theta}$$

and the factors Λ, τ are given by $\Lambda = \partial Z + \Theta\, \partial\Theta$, $\tau = \partial\Theta$. The structure relations for e^Z and e^Θ imply that the coefficients 'H' depend only on two independent ones, namely $H_{\bar\theta}{}^z$ and $H_\theta{}^z$ and they imply that the integrating factor Λ is the only independent one, the factor τ depending on it and on the 'H'. The superfield $H_{\bar\theta}{}^z$ admits a component field expansion

$$H_{\bar\theta}{}^z = \sigma_{\bar\theta}{}^z + \theta v^z + \bar\theta \mu_{\bar z}{}^z + \theta\bar\theta[-\mathrm{i}\alpha_{\bar z}{}^\theta] \ ,$$

where $\mu_{\bar{z}}{}^{z}$ and $\alpha_{\bar{z}}{}^{\theta}$ denote, respectively, the ordinary Beltrami coefficient and its fermionic partner, the so-called Beltramino. The other space-time components of $H_{\bar{\theta}}{}^{z}$, as well as all components of the superfield $H_{\theta}{}^{z}$, represent auxiliary fields in the sense of supersymmetric field theories: these auxiliary fields vanish if the geometry is restricted by supergauge conditions of Wess-Zumino-type. Actually, it is natural to consider the restriction of the geometry defined by the equation $H_{\theta}{}^{z} = 0$ (and CC) since this condition is invariant under superconformal changes of coordinates. Yet, this constraint implies that the superdiffeomorphism group has to be restricted to the subgroup which leaves it stable [3]. At the infinitesimal level, the corresponding stability condition is given by $C^{\theta} = \frac{1}{2} D C^{z}$ (and CC) where the superfields

$$\begin{aligned} C^{z} &\equiv \Xi^{z} + \Xi^{\bar{z}} H_{\bar{z}}{}^{z} + \Xi^{\theta} H_{\theta}{}^{z} + \Xi^{\bar{\theta}} H_{\bar{\theta}}{}^{z} \\ C^{\theta} &\equiv \Xi^{\theta} H_{\theta}{}^{\theta} + \Xi^{\bar{z}} H_{\bar{z}}{}^{\theta} + \Xi^{\bar{\theta}} H_{\bar{\theta}}{}^{\theta} \end{aligned}$$

parametrize infinitesimal superdiffeomorphisms generated by the supervector field $\Xi \cdot \partial \equiv \Xi^{z} \partial_{z} + \Xi^{\bar{z}} \partial_{\bar{z}} + \Xi^{\theta} D_{\theta} + \Xi^{\bar{\theta}} D_{\bar{\theta}}$, the latter being a function of the coordinates $(z, \bar{z}, \theta, \bar{\theta})$. For $H_{\theta}{}^{z} = 0$, the differential equation satisfied by the integrating factor Λ takes the simple form

$$[\bar{D} - H_{\bar{\theta}}{}^{z} \partial + \frac{1}{2} (D H_{\bar{\theta}}{}^{z}) D] \Lambda = (\partial H_{\bar{\theta}}{}^{z}) \Lambda \ .$$

Transformation laws and holomorphic factorization. For $H_{\theta}{}^{z} = 0$, the transformation law of $H_{\bar{\theta}}{}^{z}$ under a superconformal change of coordinates $\mathbf{z} \to \mathbf{z}'(\mathbf{z})$ is given by $(H_{\bar{\theta}}{}^{z})' = (\bar{D} \bar{\theta}')^{-1} (D\theta')^{2} H_{\bar{\theta}}{}^{z}$. Moreover, the transformation law of $H_{\bar{\theta}}{}^{z}$ under infinitesimal superdiffeomorphisms reads as [3,4]

$$\delta H_{\bar{\theta}}{}^{z} = [\bar{D} - H_{\bar{\theta}}{}^{z} \partial + \frac{1}{2} (D H_{\bar{\theta}}{}^{z}) D + (\partial H_{\bar{\theta}}{}^{z}] C^{z} \ .$$

In the Wess-Zumino supergauge, the induced variations of the ordinary Beltrami differential and of its fermionic partner take the form

$$\begin{aligned} \delta\mu &= [\bar{\partial} - \mu\partial + (\partial\mu)] c + \frac{1}{2} \alpha\epsilon \\ \delta\alpha &= [\bar{\partial} - \mu\partial + \frac{1}{2} (\partial\mu)] \epsilon + c \partial\alpha - \frac{1}{2} \alpha \partial c \ , \end{aligned}$$

where $c^{z} \equiv \xi^{z} + \xi^{\bar{z}} \mu_{\bar{z}}{}^{z}$ parametrizes infinitesimal diffeomorphisms and $\epsilon^{\theta} \equiv \xi^{\theta} + \xi^{\bar{z}} \alpha_{\bar{z}}{}^{\theta}$ local supersymmetry transformations on the underlying Riemann surface. Thus, the chosen parametrization makes the property of holomorphic factorization [5] manifest, both at the level of superfields and component fields.

Superconformal classes of vielbeins. The Beltrami parametrization of the metric on a Riemann surface can be rewritten in terms of orthonormal frame fields (the so-called zweibein forms) and these expressions can be extended to superspace [4]. Thus, one writes the super zweibein forms in terms of superconformal factors (transforming under super Weyl transformations) and Beltrami superfields 'H' which parametrize superconformal classes of zweibeins. This approach to super Beltrami differentials leads to the same results as the parametrization of supercomplex structures presented above, but it is less economical due to the fact that one has to deal with super Weyl transformations.

3.2 SUPER DERIVATIVES AND CONNECTIONS

General references : [6]

For a $N = 1$ SRS $\mathbf{S\Sigma}$, it is convenient to denote the Jacobian of the superconformal change of coordinates $\mathbf{z} \to \mathbf{z}'(\mathbf{z})$ by $\mathrm{e}^{-w} \equiv D\theta'$.

Superconformal field. Let n be an integer. Then, a superconformal field of weight $\frac{n}{2}$ on $\mathbf{S\Sigma}$ is a superfield with the transformation property $C_n'(\mathbf{z}', \bar{\mathbf{z}}') = \mathrm{e}^{nw} C_n(\mathbf{z}, \bar{\mathbf{z}})$ with respect to a superconformal change of coordinates [1,8]. The field C_n is taken to have Grassmann parity $(-)^n$ and the space of these fields is denoted by $\mathcal{F}_n$.

Superaffine connection. A superaffine connection on $\mathbf{S\Sigma}$ is a collection $\{\Gamma_\theta(\mathbf{z}, \bar{\mathbf{z}})\}$ of odd superfields which are locally superanalytic (i.e. $\bar{D}\Gamma_\theta = 0$) and which transform under a superconformal change of coordinates as

$$\Gamma_{\theta'}(\mathbf{z}') = \mathrm{e}^{w}[\Gamma_\theta(\mathbf{z}) - D_\theta w] \ .$$

Supercovariant derivative. Given a superaffine connection, one can locally define a supercovariant derivative which maps superconformal fields to superconformal fields:

$$\begin{array}{rcccl} \nabla_{(n)} \ : \ \mathcal{F}_n & \longrightarrow & \mathcal{F}_{n+1} & & \\ C_n & \longmapsto & \nabla_{(n)} C_n & \equiv & (D_\theta + n\,\Gamma_\theta) C_n \ . \end{array}$$

Super Schwarzian derivative. The super Schwarzian derivative of the coordinate transformation $\mathbf{z} \to \mathbf{z}'(\mathbf{z})$ is defined by [1,8]

$$\begin{aligned} \mathcal{S}(\mathbf{z}', \mathbf{z}) &= -[D^3 w + (\partial w)(Dw)] \\ &= \frac{\partial^2 \theta'}{D\theta'} - 2\frac{(\partial\theta')(D^3\theta')}{(D\theta')^2} \ . \end{aligned}$$

Under the composition of superconformal transformations, $\mathbf{z} \to \mathbf{z}' \to \mathbf{z}''$, it transforms according to the chain rule

$$\mathcal{S}(\mathbf{z}'', \mathbf{z}) = \mathrm{e}^{-3w} \mathcal{S}(\mathbf{z}'', \mathbf{z}') + \mathcal{S}(\mathbf{z}', \mathbf{z}) \ .$$

Superprojective connection. A superprojective (or super Schwarzian) connection on $\mathbf{S\Sigma}$ is an expression $\mathcal{R}$ that is locally given by a collection of odd superfields $\mathcal{R}_{z\theta}$ which are locally superanalytic (i.e. $\bar{D}\mathcal{R}_{z\theta} = 0$) and which transform under a superconformal change of coordinates according to

$$\mathcal{R}_{z'\theta'}(\mathbf{z}') = \mathrm{e}^{3w}[\mathcal{R}_{z\theta}(\mathbf{z}) - \mathcal{S}(\mathbf{z}', \mathbf{z})] \ .$$

From $\bar{D}^2 = \bar{\partial}$ and $\bar{D}\mathcal{R}_{z\theta} = 0$, it follows that $\bar{\partial}\mathcal{R}_{z\theta} = 0$. Thus, the superfield $\mathcal{R}_{z\theta}$ admits a θ-expansion of the form

$$\mathcal{R}_{z\theta}(z, \theta) = \frac{\mathrm{i}}{2}\rho_{z\theta}(z) + \theta[\frac{1}{2}R_{zz}(z)]$$

and the transformation law of $\mathcal{R}_{z\theta}$ implies that its component R_{zz} transforms like an ordinary projective connection up to supersymmetric contributions. The difference of two superprojective connections is a superanalytic quadratic differential and therefore these connections define a superaffine space; the space of equivalence classes of such connections has dimension $3g - 3|2g - 2$.

If Γ is a superaffine connection, then $\mathcal{R}_{z\theta} \equiv -[\partial\Gamma_\theta + \Gamma_\theta(D\Gamma_\theta)]$ represents a superprojective connection. The last equation is formally identical to the super Miura transformation which is relevant for the study of super KdV equations.

3.3 SUPERPROJECTIVE STRUCTURES

Superprojective transformation. Coordinates $\mathbf{Z} = (Z, \Theta)$ belonging to a superprojective atlas of a SRS are related to each other by superprojective (super Möbius) transformations [1,8]. These are superconformal changes of coordinates $\mathbf{Z} \to \mathbf{Z}'(\mathbf{Z})$ for which the super Schwarzian derivative vanishes, i.e. $\mathcal{S}(\mathbf{Z}', \mathbf{Z}) = 0$. Direct integration of this equation and of the superconformal condition $D_\Theta Z' = \Theta'(D_\Theta\Theta')$ gives [9]

$$\begin{aligned} Z' &= \frac{aZ + b}{cZ + d} + \Theta\frac{\gamma Z + \delta}{(cZ + d)^2} \\ \Theta' &= \frac{\gamma Z + \delta}{cZ + d} + \Theta\frac{1 + \frac{1}{2}\delta\gamma}{cZ + d} \end{aligned}$$

with $ad - bc = 1$. Here, a, b, c, d are even and γ, δ odd constants; the parameters have been redefined in such a way that the even part of the transformation for Z coincides with ordinary projective transformations. The associated Jacobian then reads $D_\Theta\Theta' = (\tilde{c}Z + \tilde{d} + \Theta\tilde{\gamma})^{-1}$ with $\tilde{c} = c(1 - \frac{1}{2}\delta\gamma)$, $\tilde{d} = d(1 - \frac{1}{2}\delta\gamma)$, $\tilde{\gamma} = c\delta - d\gamma$.

Superprojective structure. On a compact SRS, there is a one-to-one correspondence between superprojective connections and superprojective structures (i.e. superprojective atlases) [6]. This relation is expressed by $\mathcal{R}_{z\theta}(\mathbf{z}) = \mathcal{S}(\mathbf{Z}, \mathbf{z})$ where $\mathbf{Z}$ belongs to a superprojective coordinate system and $\mathbf{z}$ to a generic one. In fact, the so-defined expression transforms correctly with respect to a superconformal change of $\mathbf{z}$ and it is inert under a super Möbius transformation of $\mathbf{Z}$.

Quasi-primary superfield. A superfield $\mathcal{C}_n(\mathbf{Z})$ (with n integer) transforming covariantly with respect to superprojective changes of coordinates, $\mathcal{C}'_n(\mathbf{Z}') = (D_\Theta \Theta')^{-n}\, \mathcal{C}_n(\mathbf{Z})$, is called a quasi-primary superfield of weight $\frac{n}{2}$ [1,8,9].

3.4 SUPERCOVARIANT OPERATORS

General references : [9,10]

Superconformally covariant operator. The most general superconformally covariant differential operator (super CCO) of order $2n+1$ acting on superconformal fields may be cast into the form [9,10,11]

$$\mathcal{L}^{(n)} = D^{2n+1} + a_2 D^{2n-1} + \ldots + a_{2n+1} \ ,$$

where $D = \partial_\theta + \theta\partial$ and where the coefficients $a_k \equiv a_k^{(n)}(z,\theta)$ are locally analytic superfields. The coefficient a_k is even (odd) for k even (odd). Under a superconformal change of coordinates, these coefficients transform in such a way that $\mathcal{L}^{(n)} : \ \mathcal{F}_{-n} \to \mathcal{F}_{n+1}$ (where $\mathcal{F}_k$ represents the space of superconformal fields of weight $\frac{k}{2}$). More specifically, a_2 belongs to $\mathcal{F}_2$ and the combination $\tilde{a}_3 \equiv a_3 - \frac{1}{2} D a_2$ is a multiple of a superprojective connection, $\tilde{a}_3 = \frac{1}{2} n(n+1)\mathcal{R}$. The remaining coefficients a_k transform in a more complicated way.

Since $\tilde{a}_3(\mathbf{z}) \propto \mathcal{S}(\mathbf{Z}, \mathbf{z})$, this coefficient vanishes if $\mathbf{z}$ is chosen to belong to the same superprojective atlas as $\mathbf{Z} = (Z, \Theta)$. Thus, one can construct super CCO's by starting from such a superprojective atlas and defining simple operators which are covariant with respect to super Möbius transformations, and then going over to a generic coordinate system.

Super Bol operator. The simplest super Möbius covariant differential operator is given by $(D_\Theta)^{2n+1}$. Upon passage to a generic coordinate system, it yields the super Bol operator $\mathcal{L}_n$, the first few examples being given by $\mathcal{L}_0 = D$ and

$$\begin{aligned} \mathcal{L}_1 &= D^3 + \mathcal{R} \\ \mathcal{L}_2 &= D^5 + 3\mathcal{R}\partial + (D\mathcal{R})D + 2(\partial\mathcal{R}) \ . \end{aligned}$$

The odd superdifferential operator $\mathcal{L}_1$ acts on a superconformal field $C_{-1} \equiv C$. By applying D to $\mathcal{L}_1 C$ and subsequently projecting onto the lowest component of the resulting superfield, one finds

$$
\begin{aligned}
(D\mathcal{L}_1 C)| &= \left[D^4 + (D\mathcal{R}_{z\theta})|\right] C| - \mathcal{R}_{z\theta}|(D\mathcal{C})| \\
&= [\partial^2 + \frac{1}{2}R_{zz}]c + \rho_{z\theta}(DC)| \quad ,
\end{aligned}
$$

i.e. the basic Bol operator L_2 plus a fermionic contribution.

Operators parametrized by superconformal fields. Along the same procedure, one can introduce covariant operators $M^{(n)}_{W_k}$ of order $2n+1-k$ depending on a superprojective structure and on superconformal fields W_k, the simplest expressions being given by

$$
\begin{aligned}
M^{(n)}_{W_{2n+1}} &= W_{2n+1} \ , \quad M^{(n)}_{W_{2n}} = W_{2n} D + \frac{1}{2}(DW_{2n}) \\
M^{(n)}_{W_{2n-1}} &= W_{2n-1}\partial + \frac{1}{2n-1}(DW_{2n-1})D + \frac{n}{2n-1}(\partial W_{2n-1}) \ .
\end{aligned}
$$

Classification theorem. By adding the operators $M^{(n)}_{W_k}$ (with $1 \leq k \leq 2n+1$) to the super Bol operator $\mathcal{L}_n$, one obtains again a CCO of order $2n+1$:

$$
\mathcal{L} = \mathcal{L}_n + M^{(n)}_{W_1} + \ldots + M^{(n)}_{W_{2n+1}} \ .
$$

Any operator of order $2n+1$ which is superconformally covariant can be cast into this form (with $W_1 = 0 = W_3$) according to the classification theorem for super CCO's: For a superconformally covariant operator of order $2n+1$, one can find a reparametrization of the coefficients a_k in terms of a superprojective connection $\mathcal{R}$ and $2n+1$ superconformal fields $W_1, \ldots, W_{2n+1}$ (which are differential polynomials in the a_k).

3.5 SUPER W-ALGEBRAS

Matrix representation of super CCO's. The scalar, superconformally covariant differential equation $0 = \mathcal{L}_1 F_3 \equiv [D^3 + \mathcal{R}]F_3$ can be rewritten as a system of three first-order differential equations:

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} D & 0 & \mathcal{R} \\ -1 & D & 0 \\ 0 & -1 & D \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}
$$

The latter equation has the form $\vec{0} = (D - \mathcal{A}_\theta)\vec{F}$ with a matrix $\mathcal{A}_\theta$ which takes its values in the Lie superalgebra $sl(2|1)$. One can pair the connection component $\mathcal{A}_\theta$ with components $\mathcal{A}_{\bar{\theta}}$ and $\mathcal{A}_z, \mathcal{A}_{\bar{z}}$ so as to obtain a superconnection with values in $sl(2|1)$ (or $sl(n{+}1|n)$ in the generic case). By imposing a zero curvature condition on this superconnection, one can construct the $N=2$ super W_3 algebra [10].

As a matter of fact, the graded matrix $\mathcal{A}_\theta$ does not have the standard matrix format which consists of arranging the even and odd matrix elements into blocks: it rather represents an example of a nonstandard matrix format, namely the diagonal format for which there are alternatively even and odd diagonals [12]. Such nonstandard matrix formats naturally occur in various physical applications, e.g. in superconformal field theory, superintegrable models, for super W-algebras and quantum supergroups.

Super W_n algebra. The classical super W_n algebra represents a non-linear extension of the classical super Virasoro algebra. It is generated by the super stress tensor $\mathcal{T}$ and superconformal fields $W_2, \dots$ which correspond to the currents for the W symmetries. The super stress tensor $\mathcal{T} \equiv \frac{1}{2}n(n+1)\mathcal{R}$ (which transforms like a superprojective connection $\mathcal{R}$) and the superconformal fields W_k parametrize the most general superconformally covariant operator $\mathcal{L}^{(n)}$ of order $2n+1$. In the simplest case ($n=1$), one has $\mathcal{L}^{(1)} = D^3 + a_2^{(1)}D + a_3^{(1)}$. The Poisson brackets for the super stress tensor $\mathcal{T} \equiv \mathcal{R} = a_3^{(1)} - \frac{1}{2}Da_2^{(1)}$ and for the superconformal field $W_2 = a_2^{(1)} \equiv \mathcal{J}$ then take the form [9,10,13,14]

$$
\begin{aligned}
\{\mathcal{T}(\mathbf{z}'), \mathcal{T}(\mathbf{z})\} &= \frac{1}{2}\mathcal{L}_2\delta(\mathbf{z}-\mathbf{z}') \\
\{\mathcal{J}(\mathbf{z}'), \mathcal{J}(\mathbf{z})\} &= 2\mathcal{L}_1\delta(\mathbf{z}-\mathbf{z}') \\
\{\mathcal{T}(\mathbf{z}'), \mathcal{J}(\mathbf{z})\} &= -[(\partial\mathcal{J}) + \mathcal{J}\partial - \frac{1}{2}(D\mathcal{J})D]\delta(\mathbf{z}-\mathbf{z}') \ ,
\end{aligned}
$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the super Bol operators. The Poisson brackets of the first line define the super Virasoro algebra while the last relation represents the transformation law of the superconformal field $\mathcal{J}$ under a superconformal change of coordinates generated by the stress tensor $\mathcal{T}$.

3.6 $N = \frac{1}{2}$ THEORY

In many respects, the so-called $N = \frac{1}{2}$ or $(1,0)$ geometry is similar to the $N=1$ or $(1,1)$ geometry [15,3]. Both of these theories represent particular cases of the so-called (p,q) supersymmetry [16]. $(1,0)$ superspace is locally parametrized by coordinates $(z, \bar{z}, \theta)$ where θ denotes an anticommuting variable. The theory is asymmetric with respect to the operation of complex conjugation and the proper way to look at it, is to consider the Minkowskian set-up where z and $\bar{z}$ represent real light-cone coordinates $x_+ = x + ct$, $x_- = x - ct$ and θ a real Grassmanian variable. This allows to give a well-defined meaning to superconformal coordinate transformations and it leads to real action functionals for the dynamical fields. In fact, a $(1,0)$ supersymmetric field theory involves chiral fermions and therefore the action cannot be made real in the Euclidean framework (the problem being very analogous to the one encountered in a four dimensional Euclidean theory containing Majorana fermions).

Superconformal transformation. A change of local coordinates $(z, \bar{z}, \theta) \to (z', \bar{z}', \theta')$ is said to be superconformal if it satisfies $z' = z'(z, \theta)$, $\theta' = \theta'(z, \theta)$, $\bar{z}' = \bar{z}'(\bar{z})$ as well as the superconformal condition $Dz' = \theta'(D\theta')$ (where $D \equiv \partial_\theta + \theta\partial$) [3]. Under such a coordinate transformation, the canonical super tangent space vectors D and $\partial_{\bar{z}}$ change according to $D = (D\theta')D'$ and $\partial_{\bar{z}} = (\partial_{\bar{z}}\bar{z}')\partial_{\bar{z}'}$.

Super Beltrami differentials. The $(1,0)$ superconformal structures are parametrized by two odd and one even Beltrami superfields, $H_\theta{}^{\bar{z}}, H_\theta{}^z$ and $H_{\bar{z}}{}^z$ and the geometry can be constrained by the condition $H_\theta{}^z = 0$ [15,3]. The basic superfields have the form

$$\begin{aligned} H_{\bar{z}}{}^z &= \mu_{\bar{z}}{}^z + \theta[\mathrm{i}\alpha_{\bar{z}}{}^\theta] \\ H_\theta{}^{\bar{z}} &= \rho_\theta{}^{\bar{z}} + \theta\mu_z{}^{\bar{z}} \end{aligned}$$

and contain the Beltrami coefficients $\mu_{\bar{z}}{}^z$, $\mu_z{}^{\bar{z}}$ and their fermionic partner, the chiral Beltramino $\alpha_{\bar{z}}{}^\theta$. In the Wess-Zumino supergauge, these are the only variables to survive. As in the $(1,1)$ supersymmetric theory, the property of holomorphic factorization can be realized in a manifest way [3].

3.7 $N = 2$ THEORY

A $N = 2$ super Riemann surface is locally parametrized by an even complex coordinate z and two independent odd complex coordinates θ and $\bar{\theta}$ [17,18]. (The complex conjugate variables are then denoted by z^*, θ^* and $\bar{\theta}^*$.) There is a new feature in $N = 2$ superspace geometry which makes this theory considerably richer and more complicated than the $N = 1$ theory: the "square root" of the translation generator ∂ is not given by a single odd operator as in $N = 1$ supersymmetry (where $D^2 = \partial$), but it involves two odd operators,

$$D = \frac{\partial}{\partial\theta} + \frac{1}{2}\bar{\theta}\partial \ , \quad \bar{D} = \frac{\partial}{\partial\bar{\theta}} + \frac{1}{2}\theta\partial \ ,$$

satisfying $\{D, \bar{D}\} = \partial$ (and $D^2 = 0 = \bar{D}^2$). Therefore, one has to deal with partial differential equations (involving D and $\bar{D}$) rather than ordinary differential equations (only involving D). Another aspect of the algebra $\{D, \bar{D}\} = \partial$ consists of the fact that it introduces a $U(1)$ symmetry into the theory: after projection from the super Riemann surface to the underlying ordinary Riemann surface, one thereby recovers $U(1)$-transformations in addition to the familiar conformal transformations. Yet, the usual geometric structures existing on ordinary Riemann surfaces or on $N = 1$ super Riemann surfaces can be generalized to the $N = 2$ case, e.g. Beltrami differentials [18,19], conformal fields [17,18], affine and projective connections [20] and conformally covariant operators [20]. For a particular class of $N = 2$ superconformally covariant operators, the so-called 'sandwich operators' (relating the chiral and anti-chiral subspaces of superconformal fields), one can give a matrix representation [20]. The latter allows to construct $N = 2$ super W_n-algebras [14] from a zero curvature condition.

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